

# An Arithmetic Metric

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## Abstract

What is the distance between 11 (a prime number) and 12 (a highly composite number)? If your answer is 1, then ask yourself is this reasonable? In this work, we will introduce a distance between natural numbers based on their arithmetic properties, instead of their position on the real line.

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## 1 Introduction

When the concepts of distance and metric space are introduced in a standard advanced calculus course, it is customary to present some examples of metrics. These usually consist of the absolute value (for  $\mathbb{R}$ ), the  $l_p$  norms

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(for  $\mathbb{R}^n$  and  $\mathbb{R}^{\mathbb{N}}$ ) and the  $L_p$  norms (for  $\mathbb{R}^{\mathbb{R}}$ ) [3]. In most courses, the only "exotic" metric that students learn about is the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

For those students, whose main interest is algebra, these examples seem to imply that the theory of metric spaces is something that they should not care about (except for the brief moment when they need to pass the required course!).

The objective of this article is to provide a non-trivial example of a metric that should be interesting to algebraists and analysts alike. It should also appeal to those interested in graph theory and discrete mathematics.

To motivate our definition, let's consider the following question: What is the distance between 11 and 12? As real numbers, the answer is of course  $d(11, 12) = |12 - 11| = 1$ . However, if we take into account their arithmetic properties, they are very different numbers indeed. While 11 is a prime number, 12 is a highly composite number, i.e., it has more divisors than any smaller natural number. Thus, it seems that the distance between them as natural numbers should be based on divisibility rather than on their location on the real line.

We can look at the problem from a slightly different perspective, if we consider the *Hasse diagram* of the set  $I_{12} = \{1, 2, \dots, 12\}$ , i.e., the graph formed with numbers  $1, 2, \dots, 12$  as vertices and edges connecting two numbers  $a < b$  iff  $a|b$  (see Figure 1). If we define the distance between two numbers in the Hasse diagram as the number of edges in a shortest path connecting them, then clearly we have  $d(11, 12) = 4$ . This result seems more satisfactory than the previous calculation using the absolute value.

If we carefully examine the Hasse diagram, we conclude that our proposed distance should have the following properties:

1. If  $a < b$ , then

$$d(a, b) = 1 \Leftrightarrow \exists p \in \mathbb{P} \text{ such that } b = ap, \quad (1)$$

where

$$\mathbb{P} = \{p \in \mathbb{N} \mid p \text{ is a prime number}\}.$$

In other words, the only way of advancing from one number to another 1 unit of distance is by multiplying the number by a prime.

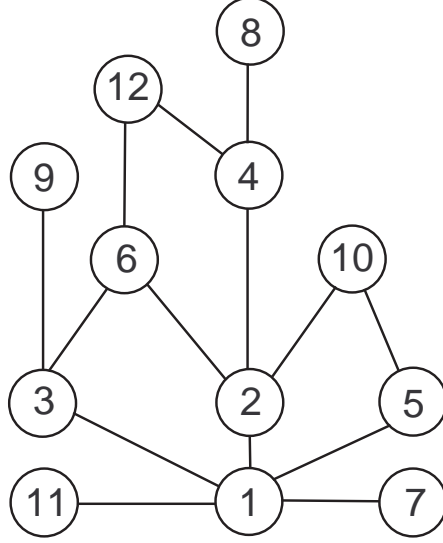


Figure 1: The Hasse diagram of the set  $I_{12}$ .

2. If  $l = \text{lcm}(a, b)$  and  $g = \text{gcd}(a, b)$ , then

$$d(a, l) + d(l, b) = d(a, g) + d(g, b), \quad (2)$$

which says that the distance between  $a$  and  $b$  going through  $\text{lcm}(a, b)$  is the same as going through  $\text{gcd}(a, b)$  (see Figure 2).

In the following section, we will define  $d(a, b)$  precisely and prove that it satisfies 1 and 2.

## 2 Main result

We begin by reviewing some standard notations.

**Definition 1** If  $n \in \mathbb{N}$  and  $p \in \mathbb{P}$ , we define,  $\nu_p(n)$ , the  $p$ -adic valuation of  $n$ , by

$$\nu_p(n) = \max \{k \in \mathbb{N}_0 \mid p^k | n\},$$

where  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . It follows from the fundamental theorem of arithmetic [5] that

$$n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}.$$

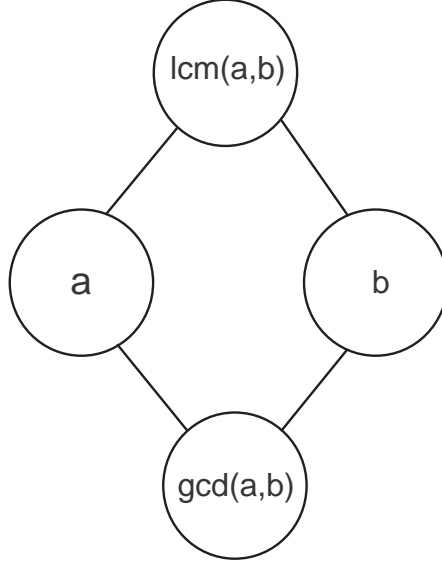


Figure 2: The Hasse diagram of the set  $\{a, b, \gcd(a, b), \text{lcm}(a, b)\}$ .

We define

$$\Omega(n) = \sum_{p \in \mathbb{P}} \nu_p(n). \quad (3)$$

The function  $\Omega(n)$  is called (surprise!) the *Big Omega function* [1, p. 354]. It represents the total number of prime factors of  $n$ , counting prime factors with multiplicity. The following lemma states that  $\Omega(n)$  is totally additive.

**Lemma 2** *If  $a, b \in \mathbb{N}$ , then*

$$\Omega(ab) = \Omega(a) + \Omega(b). \quad (4)$$

We have now all the necessary elements to define our distance. We denote by  $\mathbb{N}_0$  the set  $\mathbb{N} \cup \{0\}$ .

**Definition 3** *If  $a, b \in \mathbb{N}$ , we define the function  $d : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}_0$  by*

$$d(a, b) = \Omega[\text{lcm}(a, b)] - \Omega[\gcd(a, b)]. \quad (5)$$

Although possible, it is a bit complicated to prove that  $d(a, b)$  is a distance using the definition (5). The following theorem gives an alternative representation for  $d(a, b)$ , from which it is clear that  $d(a, b)$  is indeed a metric.

**Theorem 4** *If  $a, b \in \mathbb{N}$ , then*

$$d(a, b) = \sum_{p \in \mathbb{P}} |\nu_p(a) - \nu_p(b)|. \quad (6)$$

**Proof.** Since [5]

$$\text{lcm}(a, b) = \prod_{p \in \mathbb{P}} p^{\max\{\nu_p(a), \nu_p(b)\}}, \quad \text{gcd}(a, b) = \prod_{p \in \mathbb{P}} p^{\min\{\nu_p(a), \nu_p(b)\}},$$

then

$$d(a, b) = \sum_{p \in \mathbb{P}} [\max\{\nu_p(a), \nu_p(b)\} - \min\{\nu_p(a), \nu_p(b)\}].$$

But for any real numbers  $x, y$

$$\max\{x, y\} - \min\{x, y\} = \begin{cases} x - y, & x \geq y \\ y - x, & x \leq y \end{cases} = |x - y|$$

and the result follows. ■

**Corollary 5**  *$(\mathbb{N}, d)$  is a metric space.*

Using the metric  $d$ , we can give a nice topological interpretation to the set of prime numbers  $\mathbb{P}$ .

**Example 6** *If we denote by  $\overline{B}_r(x)$  the closed ball of radius  $r$  centered at  $x$ , i.e.,*

$$\overline{B}_r(x) = \{y \mid d(x, y) \leq r\},$$

*we have*

$$\overline{B}_1(1) = \mathbb{P}.$$

Just as the absolute value is a *translation invariant metric*, i.e.,

$$|(x + z) - (y + z)| = |x - y|,$$

the distance  $d(a, b)$  is a *multiplicative invariant metric*.

**Proposition 7** *If  $a, b, c \in \mathbb{N}$ , then*

$$d(ac, bc) = d(a, b). \quad (7)$$

**Proof.** We have [4]

$$\text{lcm}(ac, bc) = c \text{lcm}(a, b) \quad \text{and} \quad \text{gcd}(ac, bc) = c \text{gcd}(a, b).$$

Thus, from (4) we conclude that

$$\Omega[\text{lcm}(ac, bc)] - \Omega[\text{gcd}(ac, bc)] = \Omega[\text{lcm}(a, b)] - \Omega[\text{gcd}(a, b)]$$

and the result follows. ■

We should now check that  $d$  satisfies the properties (1) and (2).

**Theorem 8** *If  $a, b \in \mathbb{N}$  and  $a < b$ , then*

$$d(a, b) = 1 \Leftrightarrow \exists p \in \mathbb{P} \text{ such that } b = ap.$$

**Proof.** It is clear that

$$|\nu_p(a) - \nu_p(b)| \in \mathbb{N}_0, \quad \forall p \in \mathbb{P}.$$

Thus, from (6) we have

$$1 = d(a, b) = \sum_{p \in \mathbb{P}} |\nu_p(a) - \nu_p(b)|$$

if and only if  $\exists p \in \mathbb{P}$  such that  $|\nu_p(a) - \nu_p(b)| = 1$  and

$$|\nu_q(a) - \nu_q(b)| = 0 \quad \forall q \in \mathbb{P} \setminus \{p\}.$$

Since  $a < b$ , we conclude that

$$\nu_p(b) = \nu_p(a) + 1 \quad \text{and} \quad \nu_q(a) = \nu_q(b) \quad \forall q \in \mathbb{P} \setminus \{p\}$$

or, equivalently,  $b = ap$ . ■

**Theorem 9** *If  $a, b \in \mathbb{N}$ ,  $l = \text{lcm}(a, b)$  and  $g = \text{gcd}(a, b)$ , then*

$$d(a, l) + d(l, b) = d(a, b) = d(a, g) + d(g, b).$$

**Proof.** We have

$$\max \{\nu_p(a), \nu_p(l)\} = \max \{\nu_p(a), \max \{\nu_p(a), \nu_p(b)\}\} = \max \{\nu_p(a), \nu_p(b)\}$$

and

$$\min \{\nu_p(a), \nu_p(l)\} = \min \{\nu_p(a), \max \{\nu_p(a), \nu_p(b)\}\} = \nu_p(a).$$

Hence,

$$d(a, l) + d(l, b) = 2\Omega[\text{lcm}(a, b)] - \Omega(a) - \Omega(b). \quad (8)$$

Using [4]

$$ab = \text{lcm}(a, b) \text{gcd}(a, b) \quad (9)$$

and (4) in (8), we obtain

$$d(a, l) + d(l, b) = \Omega[\text{lcm}(a, b)] - \Omega[\text{gcd}(a, b)] = d(a, b).$$

Since

$$\max \{\nu_p(a), \nu_p(g)\} = \max \{\nu_p(a), \min \{\nu_p(a), \nu_p(b)\}\} = \nu_p(a)$$

and

$$\min \{\nu_p(a), \nu_p(g)\} = \min \{\nu_p(a), \min \{\nu_p(a), \nu_p(b)\}\} = \min \{\nu_p(a), \nu_p(b)\},$$

then

$$d(a, g) + d(g, b) = \Omega(a) + \Omega(b) - 2\Omega[\text{gcd}(a, b)]. \quad (10)$$

Using (4) and (9) in (8), the result follows. ■

The number of elements in the set  $S_k = \{m \in \mathbb{N} \mid \Omega(m) = k\}$  is clearly infinite. A more interesting question would be to describe the number of elements in the set  $S_k \cap I_n$  as  $n \rightarrow \infty$ . We have [1, 22.18]

$$\#(S_k \cap I_n) \sim \frac{n}{\ln(n)} \frac{[\ln \ln(n)]^{k-1}}{(k-1)!}, \quad n \rightarrow \infty, \quad (11)$$

where  $\#$  represents cardinality and  $a \sim b$  means that  $\frac{a}{b} \rightarrow 1$  as  $n \rightarrow \infty$ . The proof of (11) is beyond the reach of this paper, since it contains (or depends on) a proof of the Prime Number Theorem [2].

What we can do instead is estimate the maximum distance between two numbers in the set  $I_n$ , which we will do in the next section.

## 2.1 The diameter of $I_n$

**Definition 10** If  $s \in (0, \infty)$ ,  $p \in (1, \infty)$ , let

$$\xi_p(s) = \max \{k \in \mathbb{N} \mid p^k \leq s\}. \quad (12)$$

The next couple of lemmas follow immediately from the definition of  $\xi_p(s)$ .

**Lemma 11** If  $s \in (0, \infty)$ ,  $p \in (1, \infty)$ , then

$$\xi_p(s) = \left\lfloor \frac{\ln(s)}{\ln(p)} \right\rfloor,$$

where

$$\lfloor x \rfloor = \max \{k \in \mathbb{Z} \mid k \leq x\}.$$

**Lemma 12** If  $m \in (0, \infty)$ ,  $p \in (1, \infty)$ , then

$$1. \quad \xi_q(m) \leq \xi_p(m), \quad \text{if } p \leq q. \quad (13)$$

$$2. \quad \xi_p(m) \leq \xi_p(n) \quad \text{if } m \leq n. \quad (14)$$

We can now obtain a first estimate comparing the growth of  $\Omega(n)$  and  $\xi_p(n)$ .

**Lemma 13** Let  $n \in \mathbb{N}$ . Then,

$$1. \text{ For all } n \in \mathbb{N} \quad \Omega(n) \leq \xi_2(n). \quad (15)$$

$$2. \text{ If } n \in \mathbb{N} \text{ is odd, then} \quad \Omega(n) \leq \xi_3(n). \quad (16)$$

**Proof.**

1. Since

$$2^{\Omega(n)} = \prod_{p \in \mathbb{P}} 2^{\nu_p(n)} \leq \prod_{p \in \mathbb{P}} p^{\nu_p(n)} = n,$$

the result follows from (12).



2. Similarly, if  $n$  is odd, then

$$3^{\Omega(n)} = \prod_{p \in \mathbb{P}} 3^{\nu_p(n)} \leq \prod_{p \in \mathbb{P}} p^{\nu_p(n)} = n,$$

since  $\nu_2(n) = 0$ .

■

We have now all the necessary elements to prove our result on the diameter of  $I_n$ .

**Theorem 14** *Let  $n \in \mathbb{N}$ . Then,*

$$\delta(I_n) = \xi_2(n) + \xi_3(n),$$

where

$$\delta(A) = \sup \{d(x, y) \mid x, y \in A\}$$

is the diameter of  $A$ .

**Proof.** Let  $x, y \in I_n$ . Then,

$$d(x, y) \leq d(x, 1) + d(1, y) = \Omega(x) + \Omega(y).$$

We have three possibilities:

(a) If  $x$  and  $y$  are odd numbers then, from (13), (14) and (16) we have

$$\Omega(x) + \Omega(y) \leq \xi_3(x) + \xi_3(y) \leq \xi_2(n) + \xi_3(n).$$

(b) If  $x$  or  $y$  is an odd number then, from (15), (14) and (16) we get

$$d(x, y) \leq \Omega(x) + \Omega(y) \leq \xi_2(x) + \xi_3(y) \leq \xi_2(n) + \xi_3(n). \quad (17)$$

(c) If  $x$  and  $y$  are even numbers, let

$$g = \gcd(x, y) \quad \text{and} \quad x = ag, \quad y = bg.$$

Then  $a, b \in I_n$  and  $a$  or  $b$  is an odd number. Using (7), (14) and (17) we obtain

$$d(x, y) = d(a, b) \leq \xi_2(n) + \xi_3(n).$$

Hence, we conclude that

$$\delta(I_n) \leq \xi_2(n) + \xi_3(n).$$

On the other hand, letting

$$x = 2^{\xi_2(n)}, \quad y = 3^{\xi_3(n)}$$

we have  $x, y \in I_n$  and therefore

$$\xi_2(n) + \xi_3(n) = d(x, y) \leq \delta(I_n).$$

■

In the next section, we will extend the definition of  $d$  to a bigger subset of the real numbers, which contains the rational numbers.

### 3 Extension

We remind the reader that  $l_1$  is the space of absolutely summable sequences, i.e.,

$$l_1 = \{(b_k)_{k=1}^\infty \mid \|(b_k)_{k=1}^\infty\|_1 < \infty\},$$

where  $(b_k)_{k=1}^\infty$  represents the sequence  $b_1, b_2, \dots$ , and the norm  $\|\cdot\|_1$  on  $l_1$  is defined by

$$\|(b_k)_{k=1}^\infty\|_1 = \sum_{k=1}^{\infty} |b_k|.$$

**Definition 15** *Let  $\mathbb{M}$  be defined by*

$$\mathbb{M} = \left\{ x \in \mathbb{R}^+ \mid x = \prod_{k=1}^{\infty} p_k^{\alpha_k} \text{ and } (\alpha_k \ln k)_{k=1}^\infty \in l_1 \right\},$$

where  $\mathbb{P} = \{p_1, p_2, p_3, \dots\}$ .

*With the notation above, we define  $\nu_{p_k}(x) = \alpha_k$  for  $x \in \mathbb{M}$  and*

$$\Omega(x) = \sum_{k=1}^{\infty} \nu_{p_k}(x).$$

**Remark 16** 1. Clearly  $\mathbb{Q}^+ \subset \mathbb{M}$ , but also irrational numbers like  $\sqrt[n]{a}$ , for  $a \in \mathbb{N}$ .

2. The condition  $(\alpha_k \ln k)_{k=1}^\infty \in l_1$  warrants the existence of the infinite product, since from the Prime Number Theorem [2] we have  $p_k \sim k \ln(k)$  as  $k \rightarrow \infty$ .
3. If  $x \in \mathbb{M}$ , we get

$$\Omega(x) = \sum_{k=1}^{\infty} \alpha_k < \sum_{k=1}^{\infty} |\alpha_k| \ln(k) < \infty.$$

We can now extend our definition (5).

**Definition 17** Let  $x, y \in \mathbb{M}$ . We define the distance  $d(x, y)$  by

$$d(x, y) = \sum_{p \in \mathbb{P}} |\nu_p(x) - \nu_p(y)|. \quad (18)$$

**Remark 18** If we define the function  $\Psi : \mathbb{M} \rightarrow l_1$  by

$$\Psi(x) = (\nu_{p_k}(x))_{k=1}^\infty,$$

then from (18) we see that  $\Psi$  is an isometry [3] between the metric spaces  $(\mathbb{M}, d)$  and  $(l_1, \|\cdot\|_1)$ .

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